Fluid Transmission Line Modelling Using a Variational Method

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Abstract

The modal method for time-domain modelling of pressure transients in fluid transmission lines is shown to be equivalent to a variational method. Attenuation factors, similar to the "windows" used in spectral analysis, are used to attenuate Gibbs phenomenon oscillations. Explicit formulas are given for various end conditions. The method is also applied to nonlinear damping models.
1 Introduction

Fluid power systems are most commonly simulated using networks of lumped parameter component models, that is, systems of ordinary differential equations (ODEs) and difference equations. It is therefore convenient to have lumped parameter component models for transmission lines. This requires an approximation of the governing partial differential equations (PDEs). A survey of linear PDE models for fluid transmission lines has been presented by Stecki and Davis [Stecki & Davis, 1986a,b].

Watton and Tadmori [Watton & Tadmori, 1988] compared the four most common approximations: finite differences, finite volumes ("lumping by length"), the modal method of Hullender et al. [Hsue & Hullender, 1983], [Hullender et al., 1983], and the method of characteristics. They concluded that the modal method is the most accurate, convenient, and numerically stable. A variant of the modal method was presented in [Piché & Ellman, 1995].

The modal method is originally based on a truncated series expansion of transcendental transfer functions. In this paper the modal method and its variant are rederived using variational methods. The linear transmission line models are essentially unchanged, but the variational formulation admits the modelling of non-linear PDEs. It is also the starting point for finite element methods.

The paper is organised as follows: section 2 sets up the variational formulations of the equations of motion in transmission lines. The Ritz approximation is given in section 3 using both trigonometric basis functions, yielding the modal method, and piecewise polynomial basis functions, which give the finite element method. Additional issues are discussed in section 4: approximation of the propagation operator, smoothing of Gibbs phenomenon oscillations, and correction for steady state pressure drop. Readers who are primarily interested in using the transmission line models in their simulation can skip directly to section 5, where practical implementation details are given. Extensions of the method to non-linear PDEs is briefly discussed in section 6.
2 Variational Formulation of Fluid Transmission Line Dynamics

The Laplace transformed PDEs describing the dynamics of a viscous compressible fluid in a circular transmission line are:

\[
\begin{align*}
\frac{dP(x, \bar{s})}{dx} &= -\frac{Z_0 \Gamma^2 (\bar{s})}{L \bar{s}} Q(x, \bar{s}) \\
\frac{dQ(x, \bar{s})}{dx} &= -\frac{\bar{s}}{LZ_0} P(x, \bar{s})
\end{align*}
\]  

(1a,1b)

where \( Z_0 = \frac{\rho_0 c_0}{\pi r_0^2} \) and \( \bar{s} = T s \), where \( T = L/c_0 \).

The following assumptions have been made (see [Brown, 1962], [D'Souza & Oldenberger, 1964] and [Goodson & Leonard, 1972] for detailed derivations):

- The fluid obeys Stokes' law, i.e. the fluid is Newtonian.
- The flow is laminar, i.e. the Reynolds number is 2300 or less.
- The flow is axisymmetrical. This assumption implies that the conduit is straight, although the equations can be used to model pipes with relatively small radius of curvature.
- Motion in radial direction is negligible. This implies that the longitudinal velocity component is much greater than the radial component and that pressure is constant across the cross section.
- Non-linear convective acceleration terms are negligible. The assumption is valid if velocity components are much less than the speed of sound in the fluid.
- Material properties are constants.
- The pipe walls are rigid. (The parameter representing the speed of sound can be modified appropriately to account for elasticity of the walls.)

Eliminating flow rate \( Q \) from equations (1a-b) gives the general wave with pressure \( P \) as dependent variable:

\[
-L^2 \frac{d^2 P(x)}{dx^2} + \Gamma^2 P(x) = 0 \quad x \in (0, L)
\]

(2)

Here the dependence on the normalised Laplace variable \( \bar{s} \) in equations (1a-b) is suppressed. Three different sets of boundary conditions are considered. The Neumann boundary conditions

\[
P'(0) = -\frac{Z_0 \Gamma^2}{L \bar{s}} Q_0, \quad P'(L) = \frac{Z_0 \Gamma^2}{L \bar{s}} Q_1
\]

(3)

are used when the model’s inputs are specified as the flow rates at the two ends of the line (inflow positive). The Dirichlet boundary conditions

\[
P(0) = P_0, \quad P(L) = P_1
\]

(4)

are for models with end pressure as inputs, while the Robin boundary conditions
\[ P(0) = P_0, \quad P'(L) = \frac{Z_0 \Gamma^2}{L \delta} Q_i \]  

(5)

are for models whose inputs are pressure at one end and flow at the other.

Another possible input-output combination is to consider pressure and flow rate at one end as inputs, and the corresponding values at the other end as outputs. In this "transfer matrix" approach it is easy to connect transmission lines together end to end. The resulting model is however an ill-posed boundary value problem, and the associated computational model can have serious numerical ill-conditioning. This set of b.c. is therefore not considered further.

A dual wave equation, with flow rate \( Q \) as dependent variable, can be derived by eliminating pressure \( P \) from (1a,b), yielding

\[-L^2 \frac{d^2 Q(x)}{dx^2} + \Gamma^2 Q(x) = 0 \quad x \in (0, L)\]  

(6)

Corresponding boundary conditions are Neumann:

\[ Q'(0) = -\frac{\delta}{LZ_0} P_0, \quad Q'(L) = -\frac{\delta}{LZ_0} P_1 \]  

(7)

Dirichlet:

\[ Q(0) = Q_0, \quad Q(L) = -Q_1 \]  

(8)

and Robin:

\[ Q(0) = Q_0, \quad Q'(L) = -\frac{\delta}{LZ_0} P_1 \]  

(9)

The two wave equations (2) and (6), each with three sets of boundary conditions (3-5) and (7-9), together define six different transmission line models. The variational principles for these models are now given.

The variational (or weak) formulation of the equation (2) with Neumann boundary conditions (3) is found by multiplying equation (2) by the test function \( \delta P \), integrating over the domain and integrating by parts, yielding

\[ \int_0^L \left( L^2 P' \delta P' + \Gamma^2 P \delta P \right) dx = LZ_0 \frac{\Gamma^2}{\delta} \left( Q_1 \delta P_1 + Q_0 \delta P_0 \right) \]  

(10)

The variational formulation of the problem is find the function \( P(x) \) from the Sobolev space

\[ H^1(0, L) = \{ P = P(x) \mid P, P' \in L_2(0, L) \} \]  

(11)
such that (10) holds for all \( \delta P \in H^1(0,L) \). The left side of (10) can be considered as a continuous bilinear form \( B(P, \delta P) \) from \( H^1(0,L) \times H^1(0,L) \) into \( C \). The right side of equation (10) is a linear functional \( l(\delta P) \) on \( H^1(0,L) \) that can be written

\[
l(\delta P) = LZ_0 \frac{\Gamma^2}{\delta} \langle Q(x), \delta P(x) \rangle_{\partial\Omega}, \quad \partial\Omega = \{0,L\}
\]  

(12)

where \( \langle \cdot, \cdot \rangle_{\partial\Omega} \) denotes the duality pairing between \( H^{-1/2}(\partial\Omega) \) and \( H^{1/2}(\partial\Omega) \); i.e. bilinear map of \( H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \) into \( C \).

The variational formulation of the problem is equivalent to finding the minimum of the quadratic functional

\[
I(P) = \frac{1}{2} \int_0^L \left( L^2 (P')^2 + \Gamma^2 P^2 \right) dx - LZ_0 \frac{\Gamma^2}{\delta} \left( Q_1 P_1 + Q_0 P_0 \right)
\]

\[
= \frac{1}{2} B(P, P) - l(P)
\]

(13)

over all \( P(x) \in H^1(\Omega) \). The solution of this minimization problem using the Ritz method will be described in the following section.

The derivations of the variational forms for the remaining models are similar and the results are listed here without further ado:

The variational formulation of the problem (2) with the Dirichlet boundary conditions (4) is to find \( P(x) \in H^1(0,L) \) that also satisfies the b.c. (4) such that

\[
\int_0^L \left( L^2 P' \delta P' + \Gamma^2 P \delta P \right) dx = 0, \quad \forall \delta P \in H_0^1(0,L)
\]

(14)

where the space of admissible variations \( H_0^1(0,L) \) is defined

\[
H_0^1(0,L) = \left\{ P = P(x) \mid P \in H^1(0,L), P(0) = 0, P(L) = 0 \right\}
\]

(15)

The weak formulation of the problem (2) with the Robin b.c. (5) is to find \( P(x) \in H^1(0,L) \) that satisfies the first essential b.c. \( P(0) = P_0 \) such that

\[
\int_0^L \left( L^2 P' \delta P' + \Gamma^2 P \delta P \right) dx = LZ_0 \frac{\Gamma^2}{\delta} \left( Q_1 \delta P_1 \right), \quad \forall \delta P \in H^1(0,L)
\]

(16)

where

\[
H^1(0,L) = \left\{ P = P(x) \mid P \in H^1(0,L), P(0) = 0 \right\}
\]
Similarly the variational formulation of the dual problem (6) with the Neumann b.c. (7) is to find \( Q(x) \in H^1(0, L) \) such that

\[
\int_0^L \left( \hat{L}' Q'' + \Gamma^2 Q \delta Q \right) dx = \frac{L}{Z_0} (P_1 Q_1 + P_2 Q_2), \quad \forall \delta Q \in H^1(0, L)
\]  

(17)

The weak form of problem (6) with the Dirichlet b.c. (8) is to find \( Q(x) \in H^1(0, L) \) that also satisfies both boundary conditions such that

\[
\int_0^L \left( \hat{L}' Q'' + \Gamma^2 Q \delta Q \right) dx = 0, \quad \forall \delta Q \in H^1_0(0, L)
\]  

(18)

Finally the weak form of problem (6) with the Robin b.c. (9) is to find \( Q(x) \in H^1(0, L) \) that also satisfies the essential b.c. \((Q(0) = Q_0)\) such that

\[
\int_0^L \left( \hat{L}' Q'' + \Gamma^2 Q \delta Q \right) dx = \frac{L}{Z_0} P_1 Q_1, \quad \forall \delta Q \in H^1(0, L)
\]  

(19)

3 Approximation using Ritz Method

3.1 Ritz Method

We will use the Ritz method to approximate the solution for the given problem in terms of adjustable parameters. These parameters are determined by either minimizing a functional (e.g. equation (13)) or solving the weak form of the problem (e.g. equation (10)). This kind of method is called a direct method since the approximated solutions are obtained directly from the variational form of the given problem.

Now we seek the approximate solution of problem (2) with the boundary conditions (3) that minimizes the functional

\[
I(P) = \frac{1}{2} B(P, P) - l(P)
\]  

(20)

where \( P(x) \in H^1(0, L) \). The approximate solution can be given in the form

\[
\tilde{P}(x) = \sum_{j=1}^n p_j \psi_j(x)
\]  

(21)

where \( \psi_j \in H^1(0, L) \) form a linearly independent and complete set of functions and \( p_j \), called the Ritz coefficients, are unknown parameters. Substituting the approximate solution (21) into the functional (20) that will be minimized yield conditions:
\[ \frac{\partial l(p_1, p_2, \ldots, p_n)}{\partial p_i} = 0 \quad i = 1, 2, \ldots, n \]

or

\[ \sum_{j=1}^{n} B(\psi, \psi_j)p_j - l(\psi) = 0 \quad i = 1, 2, \ldots, n \]  \hspace{1cm} (22a, 22b)

where in the latter form (22b) the symmetry of the bilinear form \( B(\cdot, \cdot) \) has been used.

Form (22b) can also be obtained directly from solving the weak problem (10):

\[ \sum_{j=1}^{n} B(\psi_j, \delta P)p_j - l(\delta P) = 0 \quad \forall \delta P \in H^1(0, L) \]  \hspace{1cm} (23)

Because we are finding the approximate solution \( \tilde{P}(x) \) of the equation (23) that is satisfied for all admissible variations \( \delta P \), it has to be satisfied for \( \delta P = \psi_i, \quad i = 1, 2, \ldots, n \). Thus we get form (22b) after using the symmetry of the bilinear form \( B(\cdot, \cdot) \).

### 3.2 Trigonometric Basis

First we will derive solution for Neumann problem (6-7) using trigonometric functions as basis \( \{\psi_j\} = \{\cos(j\pi x / L)\} \subset H^1(0, L) \) for the Ritz approximation:

\[ \tilde{P}(x) = \sum_{j=1}^{n} p_j \cos \left( \frac{j\pi x}{L} \right) \]  \hspace{1cm} (24)

Substituting approximation (24) into weak form of Neumann problem (10) we obtain equation (22b), where

\[ B(\psi, \psi_j) = \begin{cases} 
LT^2 & \text{if } j = i = 0 \\
0 & \text{if } j \neq i \\
\frac{L}{2} \left( \Gamma^2 + (i\pi)^2 \right) & \text{if } j = i \neq 0
\end{cases} \\
l(\psi_i) = \frac{LZ_0}{s} \left( (-1)^i Q_1 + Q_0 \right) \]  \hspace{1cm} (25)

Thus we have the solution for Ritz parameters \( p_i \):

\[ p_0 = \frac{Z_0}{s} (Q_1 + Q_0) \]

\[ p_i = \frac{2Z_0 \Gamma^2 / s}{\Gamma^2 + (i\pi)^2} \left( (-1)^i Q_1 + Q_0 \right) \quad i = 1, 2, \ldots, n \]  \hspace{1cm} (26)

Separating pressures in the beginning of the line \( P_0 \) and pressure in the end of line \( P_1 \) into a symmetric part \( P_s \) and an antisymmetric part \( P_a \) and making the same change of variables for flow rates \( Q_0 \) and \( Q_1 \) gives
\[ P_s = \frac{(P_0 + P_1)}{2}, \quad Q_s = \frac{(Q_0 + Q_1)}{2} \]
\[ P_a = \frac{(P_0 - P_1)}{2}, \quad Q_a = \frac{(Q_0 - Q_1)}{2} \]  

Now symmetric and antisymmetric pressures can be written in the form

\[ P_s = Z_0 H_s Q_s, \quad P_a = Z_0 H_a Q_a, \]  

where symmetric and antisymmetric transfer functions \(H_s\) and \(H_a\) are

\[ H_s = \frac{2}{\kappa} + \sum_{i=2,4,...}^{n} \frac{4 \Gamma^2 / \kappa}{\Gamma^2 + (i\pi)^2}, \quad H_a = \sum_{i=1,3,...}^{n-1} \frac{4 \Gamma^2 / \kappa}{\Gamma^2 + (i\pi)^2} \]  

The transfer functions are suitable for time simulation purposes after approximation of the propagation operator \(\Gamma^2\) in appropriate form. The Ritz method with trigonometric interpolation (24) gives the same transfer functions as in the modal approximation [Piché & Ellman, 1995], [Hullender et. al., 1983] and [Hullender & Woods, 1983] where transfer functions are derived using series theory.

Transfer functions for other weak forms can also be derived. For the weak form of the Dirichlet problem (14) we choose the basis from trigonometric functions \(\psi_j \in H^1_0(0, L)\) such that the Ritz approximation satisfies the boundary conditions

\[ \tilde{P}(x) = P_0 \frac{L-x}{L} + P_1 \frac{x}{L} + \sum_{j=1}^{n} p_j \sin \left( \frac{j\pi x}{L} \right) \]  

Solving pressure and substituting the solution into equation (1a) gives symmetric and antisymmetric flow rates

\[ Q_s = \frac{H_s P_s}{Z_0}, \quad Q_a = \frac{H_a P_a}{Z_0} \]  

where transfer functions are

\[ H_s = \sum_{i=1,3,...}^{n-1} \frac{4 \pi}{\Gamma^2 + (i\pi)^2}, \quad H_a = \frac{2}{\Gamma^2 / \kappa} + \sum_{i=2,4,...}^{n} \frac{4 \pi}{\Gamma^2 + (i\pi)^2} \]  

For the weak form of the Robin problem (16) we choose the basis from trigonometric functions \(\psi_j \in H^1_0(0, L)\) such that the Ritz approximation satisfies the essential boundary condition

\[ \tilde{P}(x) = P_0 + \sum_{j=1}^{n} p_j \sin \left( \frac{(2j-1)\pi x}{2L} \right) \]  

Substituting the Ritz approximation into the weak form (16) we obtain the Ritz parameters
\[ p_i = -\left( \frac{2}{(2i-1)\pi} P_0 + (-1)^i \frac{Z_0}{\bar{s}} Q_i \right) - \frac{2\Gamma^2}{\Gamma^2 + \left( \frac{(2i-1)\pi}{2} \right)^2} \]  

(34)

Flow rate at the beginning of the pipe \( Q_0 \) can be obtained from equation (1a) which gives

\[ Q_0 = -\frac{1}{Z_0 \Gamma^2/\bar{s}} \sum_{i=1}^{n} \frac{(2i-1)\pi}{2} p_i \]  

(35)

Similarly we can derive transfer functions for variational forms (17-19) where flow rate is dependent variable. The solution for the weak form of the Neumann problem (17) is

\[ Q_s = \frac{H_s P_i}{Z_0}, \quad Q_a = \frac{H_a P_i}{Z_0} \]  

(36)

where

\[ H_s = \sum_{i=1,3,...}^{n-1} \frac{4\bar{s}}{\Gamma^2 + (i\pi)^2}, \quad H_a = \frac{2}{\Gamma^2/\bar{s}} + \sum_{i=2,4,...}^{n} \frac{4\bar{s}}{\Gamma^2 + (i\pi)^2} \]  

(37)

This solution (36-37) is the same as the weak solution of the Dirichlet problem (31-32).

The weak pressure solution of the Dirichlet problem (18) can be obtained by first solving for flow rate from the weak form (18) and substituting it into equation (1b), yielding

\[ P_s = Z_0 H_s Q_s, \quad P_a = Z_0 H_a Q_s \]  

(38)

where symmetric and antisymmetric transfer functions \( H_s \) and \( H_a \) are

\[ H_s = \frac{2}{\bar{s}} + \sum_{i=2,4,...}^{n} \frac{4\Gamma^2/\bar{s}}{\Gamma^2 + (i\pi)^2}, \quad H_a = \sum_{i=1,3,...}^{n-1} \frac{4\Gamma^2/\bar{s}}{\Gamma^2 + (i\pi)^2} \]  

(39)

Note that solution (38-39) is the same as the solution of the Neumann problem (28-29).

Finally we will give the weak solution of the Robin problem (19). The flow rate solution is

\[ Q(x) = Q_0 + \sum_{i=1}^{n} q_i \sin \left( \frac{(2i-1)\pi x}{2L} \right) \]  

(40)

where flow rate component \( q_i \) is

\[ q_i = -\left( \frac{2}{(2i-1)\pi} P_0 + (-1)^{i+1} \frac{P_i}{Z_0 \bar{s}} \right) - \frac{2\Gamma^2}{\Gamma^2 + \left( \frac{(2i-1)\pi}{2} \right)^2} \]  

(41)
Pressure $P_0$ can be obtained by substituting the solution (41) into the equation (1b)

$$P_0 = -\frac{Z_0}{s} \sum_{i=1}^{n} \left(\frac{2i-1}{2}\right) q_i \tag{42}$$

### 3.3 Finite Element Basis

The Neumann problem (10) can also be solved using piecewise polynomials as basis functions

$$\tilde{P}(x) = \sum_{j=1}^{n} p_j \psi_j(x) \tag{43}$$

where $\psi_j \in H^1(0, L)$. The basis is chosen from linear interpolation functions

$$\psi_j(x) = \begin{cases} 
\frac{x - x_{j-1}}{x_j - x_{j-1}} & \text{if } x_{j-1} \leq x \leq x_j \\
\frac{x_{j+1} - x}{x_{j+1} - x_j} & \text{if } x_j < x \leq x_{j+1} \\
0 & \text{elsewhere}
\end{cases} \tag{44}$$

where $x_0 < 0 = x_1 < x_2 \ldots < x_n = L$. Substituting approximation (43-44) into the weak form (10) we obtain form (22b) with equally spaced nodal points

$$B(\psi_i, \psi_j) = \Gamma^2 (M)_i^j + L^2 (K)_i^j$$

$$M = \frac{l}{6} \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & 1 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 2 \end{pmatrix}, \quad K = \frac{1}{l^2} \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

$$l(\psi_i) = (f)_i, \quad f = \frac{LZ_0 \Gamma^2}{s} \begin{pmatrix} Q_0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{45}$$

Hence we have the equation (22b) in matrix form

$$(\Gamma^2 M + L^2 K)p = f \tag{46}$$

In order to include frequency dependent friction terms it is necessary to perform co-ordinate transformation $p = \Phi \eta$, where $\Phi$ is a constant transformation matrix. The columns vectors of
matrix $\Phi$ are eigenvectors such that multiplying the equation (46) by the transpose of transformation matrix $\Phi^T$ yields

$$\left(\Gamma^2 I + \Lambda\right)\eta = \Phi^T f$$

(47)

where $I$ is the identity matrix and $\Lambda$ is a diagonal matrix whose elements are eigenvalues of matrix $L^2 M^{-1} K$. Since matrix $M$ is positive definite and $K$ is positive semi-definite all eigenvalues are non-negative; one eigenvalue is zero. Solving equation (47) we obtain a modal solution that includes transfer functions $H_i$ for each eigenvalue

$$H_i = \frac{\Gamma^2 / \tilde{s}}{\Gamma^2 + \lambda_i^2}, \quad i = 1, 2, \ldots, n$$

(48)

where $\lambda_i^2$ is the $i$th element of the diagonal matrix $\Lambda$. The zero eigenvalue makes the transfer function a single integrator. The above form is useful for time simulation purposes.

Finite element solutions for other weak forms can be derived as above, but for the sake of brevity are not detailed here.

4 Modelling Details

4.1 Approximation of the Propagation Operator $\Gamma^2(\tilde{s})$

The non-dimensional propagation operator is defined as

$$\Gamma^2(\tilde{s}) = \begin{cases} \tilde{s}^2, & \text{for lossless model} \\ \tilde{s}^2 + \varepsilon \tilde{s}, & \text{for linear friction model} \\ \frac{\tilde{s}^2}{1 - \frac{2 J_1(\kappa)}{\kappa J_0(\kappa)}}, & \text{for dissipative model} \end{cases}$$

(49a,b,c)

where $\kappa^2 = -8\tilde{s}/\varepsilon$ and $\varepsilon = \frac{8 v_p L}{c_0}$.

The dissipative two-dimensional viscous compressible flow model is the only model that includes a frequency dependent friction effect. The Bessel functions of first kind $J_0$ (order zero) and $J_1$ (order one) arises from solution of Bessel's differential equation in the dissipative model. In lossless and linear friction models it is assumed that the wave is plane in cross section.

For the two-dimensional viscous compressible model, the transfer functions (29), (32), (34), (35), (37), (39) and (41) are not rational, because of the Bessel functions in the propagation operator $\Gamma^2$ in formula (49c). Approximations of the propagation operator have been proposed by many authors, [Brown, 1962], [Woods, 1983], [Yang & Tobler, 1991]. Woods' first-order square-root approximation [Woods, 1983] is accurate at low frequency as well as at high frequency. Brown's
approximation [Brown, 1962] comes from asymptotic series and it gives better results than Woods' approximation at high frequency, but results are not so accurate at low frequency.

We treat the approximation of the propagation operator differently in the denominators, which governs wave distortion and attenuation in transient responses, and in the numerator, which governs steady state responses.

In the denominators we seek an approximation of the propagation operator $\Gamma^2(\bar{s})$ that gives transfer functions in quadratic form,

$$\Gamma^2(\bar{s}) + \alpha_i^2 \approx \bar{s}^2 + \bar{s} \varepsilon_i + \omega_i^2$$

(50)

so that unknown factors are modal natural frequency $\omega_i$ and damping coefficient $\varepsilon_i$. This gives a reasonable fit to the exact transcendental transfer function.

Woods' approximation of the propagation operator $\Gamma^2(\bar{s})$ is [Woods, 1983]:

$$\Gamma^2(\bar{s}) \approx \frac{\bar{s}^2}{1 - \frac{1}{\sqrt{1 + 2\bar{s}/\varepsilon}}}$$

(51)

The resonance peaks for the transfer functions in functions (29), (32), (34), (37), (39) and (41) occur near the pole, where $\Gamma^2(\bar{s}) = -\alpha_i^2$. Here $\alpha_i$ represent correspondingly $i\pi$, $\lambda_i$ or $(2i-1)\pi/2$. Since the frequency response curve of $\Gamma^2(\bar{s})$ for small $\varepsilon$ typically has a real part numerically much larger than the imaginary part, this equation can be approximated as $\Re(\Gamma^2(\bar{s})) = -\alpha_i^2$. Applying two Newton-Raphson iterations to solve this equation gives the Puiseux series formula

$$\omega_i = \alpha_i - \frac{1}{4} \sqrt{\alpha_i \varepsilon} + \frac{1}{16} \varepsilon + O(\varepsilon^{3/2}), \quad i = 1, 2, \ldots, n$$

(52)

The modal damping is identified by the equation $\varepsilon \alpha_i = \Im(\Gamma^2(\bar{s}))$ at resonance. The first two terms of this expression's Puiseux series are

$$\varepsilon_i = \frac{1}{2} \sqrt{\alpha_i \varepsilon} + \frac{1}{16} \varepsilon + O(\varepsilon^{3/2}), \quad i = 1, 2, \ldots, n$$

(53)

Formulas (53-54) give a nearly perfect approximation of Woods' formula, which in turn gives a good approximation to the propagation operator over a much wider frequency range than the linear resistance model (Figure 1).

A different kind of approach is needed for the term $\Gamma^i(\bar{s})/\bar{s}$ that appears in formulas (29), (32), (35), (37), (39) and the term $\Gamma^2(\bar{s})$ in the numerator of equation (41). These terms can be approximated with the one-dimensional linear resistance model (49b), hence

$$\Gamma^2(\bar{s})/\bar{s} \approx \bar{s} + \varepsilon$$

(54)
In section 4.3 it is explained how it should be modified to correct the model's steady state pressure drop.

![Graph showing imaginary part of the propagation operator $\Gamma^2(\bar{\omega} = i\omega T)$ and its approximations, when $\epsilon = 1/10$.](image)

**Figure 1.** Imaginary part of the propagation operator $\Gamma^2(\bar{\omega} = i\omega T)$ and its approximations, when $\epsilon = 1/10$.

### 4.2 Linear Filtering of Gibbs Phenomenon in Modal Models

The modal transmission line models derived in the preceding sections approximate smooth solutions well with a small number of models. However, the approximation of a nonsmooth solution, such as the response to a step input, will exhibit spurious oscillation known as a Gibbs phenomenon. Increasing the number of modes does not reduce the amplitude of this oscillation, though it will tend to concentrate it into a smaller time span. The Lanczos sigma factor technique was suggested in [Piché & Ellman, 1995] to smooth the solution. Here we derive an equivalent technique based on the Ritz solution.

One way to smooth the solution is to use a weighted spatial average. For instance, a smoothed pressure solution is

$$\bar{P}(x) = \int w(x - \xi) \tilde{P}(\xi) d\xi$$

(55)

where $w(\cdot)$ is a positive weighting function. It can be thought of as the influence function of a sensor that measures pressure over a small area.

If the trigonometric basis is used in (24) or (30), and the weighted function is chosen as
\[ w(\xi) = \begin{cases} \frac{n+1}{2L} & \text{for } |\xi| \leq \frac{L}{n+1} \\ 0 & \text{otherwise} \end{cases} \quad \text{(56)} \]

then the smoothed pressure solution can be written

\[ P(x) = p_0 + \sum_{j=1}^{n} p_j w_j \psi_j(x) \quad \text{(57)} \]

where the attenuation factors \( w_j \) are

\[ w_j = \frac{\sin(j\pi/(n+1))}{j\pi/(n+1)} \quad \text{(58)} \]

If the trigonometric basis (33) is used with the function

\[ w(\xi) = \begin{cases} \frac{2n+1}{4L} & \text{for } |\xi| \leq \frac{2L}{2n+1} \\ 0 & \text{otherwise} \end{cases} \quad \text{(59)} \]

the attenuation factors are given by

\[ w_j = \frac{\sin((2j-1)\pi/(2n+1))}{(2j-1)\pi/(2n+1)} \quad \text{(60)} \]

Finally, using the weighting functions (59) to smooth the flow solution (33) gives

\[ \tilde{P}(x) = \int w(x-\xi) \tilde{P}(\xi) d\xi = p_0 + \sum_{j=1}^{n} p_j w_j \psi_j(x) \quad \text{(61)} \]

with attenuation factors given by (60).

For trigonometric basis, the smoothed solution (57), (61) differs from the original Ritz solution only by the presence of the attenuation factors \( \{w_j\} \). It is very simple to insert these scalars into a simulation model. In particular, the transfer function models (29), (32), (34), (37), (39), (41) are smoothed by inserting the attenuation factor into the summands.

The particular set of attenuation factors (58), (60) coincide with the Lanczos sigma factors [Schied, 1968] used in Fourier analysis. The digital signal processing literature [Harris, 1978] provides a variety of alternative sets of attenuation factors, or "window functions" as they are known in that discipline.

Table 1 exhibits some of the popular alternatives. The Dirichlet window corresponds to the weight function \( \delta(x) \), that is, no smoothing. The Riemann window is the piecewise constant function (56) or (59). The Hann, Hamming and Blackman-Harris windows correspond to the weight functions that are sums of equally-spaced delta functions.
Table 1. Some window functions

<table>
<thead>
<tr>
<th>The name of window function</th>
<th>expression, $w_j, j = 1,2,...,n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dirichlet</td>
<td>1</td>
</tr>
<tr>
<td>Riemann</td>
<td>$\sin \beta_j / \beta_j$</td>
</tr>
<tr>
<td>Hann</td>
<td>$(1 + \cos \beta_j) / 2$</td>
</tr>
<tr>
<td>Hamming</td>
<td>$0.54 + 0.46 \cos \beta_j$</td>
</tr>
<tr>
<td>Blackman-Harris, 3-term</td>
<td>$0.423 + 0.498 \cos \beta_j + 0.0772 \cos(2\beta_j)$</td>
</tr>
</tbody>
</table>

Coefficient $\beta_j$ in Table 1 is

$$\beta_j = \begin{cases} 
\frac{j\pi}{n+1} & \text{in (29), (32), (37) and (39)} \\
\frac{(2j-1)\pi}{2n+1} & \text{in (34) and (41)} \end{cases}$$

(62)

### 4.3 Steady State Correction

The steady state pressure drop of the original fluid line model is

$$P_0 - P_1 = \varepsilon z_0 q_a$$

(63)

In order to make the corresponding value for the finite-dimensional approximate models agree, it is necessary to make small modifications to approximation (54) of the propagation operator. These modifications are outlined below. In all cases the modified approximate propagation operator tends to the original approximation (54) as $n \to \infty$.

In the models (28-29, 38-39) the propagation operator in the numerators of the antisymmetric transfer function $H_s$ should be approximated as

$$\Gamma^2(\bar{s})/\bar{s} \approx \bar{s} + \varepsilon \sum_{i=1,3...} w_i / \omega_i^2$$

(64)

instead of (54). The symmetric transfer function $H_s$ needs no modification, since it does not participate in steady state flow.

The models (31-32, 36-37) do not need modification, since the pressure drop in determinated by the given pressures at the ends.

The model (33-35) require two modifications. The propagation operator in the numerator of (34) should be approximated as
\[ \Gamma^2(\bar{s}) \approx \bar{s}^2 + \frac{\varepsilon}{2 \sum_{i=1}^{n} w_i / \omega_i^2} \]  \hspace{1cm} (65) \]

The term \( \frac{\Gamma^2(\bar{s})}{\bar{s}} \) in (35) should be approximated by

\[ \Gamma^2(\bar{s})/\bar{s} = \bar{s} + \frac{\varepsilon \pi \sum_{j=1}^{m} (-1)^{j+1}(2j-1)w_j/\omega_j^2}{2 \sum_{i=1}^{n} w_i / \omega_i^2} \]  \hspace{1cm} (66) \]

instead of (54).

5 Implementation

5.1 Model Summary

The various approximations discussed in the foregoing are now brought together. There are three models: a Q model (with flow rates as given input), a P model (with pressures as given inputs), and a PQ model (with a pressure and a flow rate as inputs).

The Q-model is used when transmission lines are connected together or to fluid volumes (e.g. hydraulic cylinder) by components of negligible volume such as directional control valves (Fig. 2). Such connectors are modelled as orifices as in [Piché & Ellman, 1994] and [Ellman & Piché, 1996]. The Q-model transfer functions based on trigonometric basis are

\[ P_s = Z_0 H_s Q_s, \quad P_a = Z_0 H_s Q_a \]
\[ H_s \approx \frac{2}{\bar{s}} + \sum_{i=2,4,\ldots}^{n} \frac{4w_i \bar{s} + 4w_i \varepsilon}{\bar{s}^2 + \varepsilon \bar{s} + \omega_i^2}, \quad H_a \approx \sum_{i=1,3,\ldots}^{n-1} \frac{4w_i \bar{s} + 4w_i b n \varepsilon}{\bar{s}^2 + \varepsilon \bar{s} + \omega_i^2} \]  \hspace{1cm} (67) \]
\[ \alpha_i = i\pi, \quad \beta_i = \frac{i\pi}{n+1}, \quad b_a = \left(8 \sum_{i=1,3,\ldots}^{n-1} w_i / \omega_i^2\right)^{-1} \]
A pipe system with Q-pipe models. Pipes are connected by the orifice that simply calculates flow rate through it when pressure drop is known.

A 4-mode simulink realization of the Q-model is shown in Figure 3\( ^* \). Implementation in other simulation environments should be straightforward. Because the models are decoupled, the model has an obvious parallelization. For ODE-based simulators the following state space realization can be used:

\[
\begin{align*}
\dot{p}_0 &= \frac{2}{T} Z_0 Q_s \\
\dot{r}_1 &= \frac{4 w_1 b_1 \varepsilon}{T^2} Z_0 Q_s - \frac{\omega_1^2}{T^2} p_1, \\
\dot{r}_2 &= -\frac{\omega_2^2}{T^2} p_2, \\
\dot{r}_3 &= \frac{4 w_3 b_3 \varepsilon}{T^2} Z_0 Q_s - \frac{\omega_3^2}{T^2} p_3, \\
\dot{r}_4 &= -\frac{\omega_4^2}{T^2} p_4, \\
\dot{p}_1 &= \frac{4 w_1}{T} Z_0 Q_s - \frac{\varepsilon}{T} p_1 + r_1 \\
\dot{p}_2 &= \frac{4 w_2}{T} Z_0 Q_s - \frac{\varepsilon}{T} p_2 + r_2 \\
\dot{p}_3 &= \frac{4 w_3}{T} Z_0 Q_s - \frac{\varepsilon}{T} p_3 + r_3 \\
\dot{p}_4 &= \frac{4 w_4}{T} Z_0 Q_s - \frac{\varepsilon}{T} p_4 + r_4
\end{align*}
\]

Pressure in time domain at intermediate points along the transmission line can be calculated using Ritz approximation (24)

\[
P(x, t) = p_0(t) + \sum_{i=1}^{4} p_i(t) \cos \left( \frac{i \pi x}{L} \right)
\]

\( ^* \) SIMULINK realization of Q, P and PQ-models of transmission lines are available at ftp.cc.tut.fi/pub/math/piche/fluidpower
Correspondingly, flow rate in time domain at intermediate points along the transmission can be obtained from equation (1b) as

\[
Q(x) = Q_0 - \frac{T}{Z_0} \left( \dot{p}_0 \frac{x}{L} + \sum_{i=1}^{4} \dot{p}_L \sin \left( \frac{i\pi x}{L} \right) \right)
\]  

\[
\text{(70)}
\]

Figure 3. SIMULINK realisation of the Q-model with 4 modes

The P-model is used when transmission lines are connected together to fluid volumes by components of negligible resistance (Figure 4). Such connectors are modelled as hydraulic volumes [Piché & Ellman, 1996]. The P-model transfer function based on trigonometric basis is

\[
Q_s = \frac{H_s P_s}{Z_0}, \quad Q_s = \frac{H_s P_s}{Z_0}
\]

\[
H_s \approx \sum_{i=1}^{n-1} \frac{4w_j\tau}{\tau^2 + \varepsilon_i\tau + \omega_i^2}, \quad H_s \approx \frac{2}{\varepsilon + \varepsilon_i\tau + \omega_i^2} + \sum_{i=2}^{n} \frac{4w_j\tau}{\tau^2 + \varepsilon_i\tau + \omega_i^2}
\]

\[
\alpha_i = i\pi, \quad \beta_i = \frac{i\pi}{n+1}
\]  

\[
\text{(71)}
\]
Figure 4. A pipe system with P-pipe models. Pipes are connected to the volume that is modelled by an integrator. More than two pipes may be connected to a single volume.

The PQ-model is used when the connections at the ends of the transmission line are of different types. One end can be connected to an orifice and the other directly to a volume. The PQ-model transfer functions based on trigonometric basis are

\[
\begin{align*}
  p_i &= \frac{2}{(2i-1)\pi} P_0 + (-1)^i \frac{Z_0}{\bar{s}} Q_i \left[ \frac{2(\bar{s}^2 + b_1 \bar{s})}{\bar{s}^2 + \epsilon \bar{s} + \omega_i^2} \right] \\
  P_i &= P_0 + \sum_{i=1}^{n} (-1)^{i+1} p_i, \quad Q_0 = -\frac{1}{Z_0(\bar{s} + b_2 \epsilon)} \sum_{i=1}^{n} \frac{(2i-1)\pi}{2} p_i \\
  \alpha_i &= \frac{(2i-1)\pi}{2}, \quad \beta_i = \frac{(2i-1)\pi}{2n+1}, \\
  b_1 &= \left( 2 \sum_{i=1}^{n} \frac{w_i}{\omega_i^2} \right)^{-1}, \quad b_2 = \pi b_1 \sum_{i=1}^{n} (-1)^{i+1}(2i-1)w_i/\omega_i^2
\end{align*}
\] (72)

When modelling a network containing several transmission line elements, one should select the number of modes such that the ratio \( n / L \) is about the same for each element. This ratio reflects the bandwidth over which the model is supposed to be accurate. Thus, longer transmission lines require more modes to accurately model the same frequency range as shorter lines with fewer modes. Assigning too many modes to short transmission lines is inefficient, since the numerical ODE integrator will need to use small time steps to follow the high frequency modes. This numerical cost would be especially evident with simulators that use BDF methods (such as Gear's method) as the integrator, because of their known inefficiency with oscillatory problems.
5.2 Numerical Results

To examine the differences between trigonometric and FEM models consider a single pipe with the properties given in Table 3.

Table 3. The physical properties of pipe system

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of line, $L$</td>
<td>20 m</td>
</tr>
<tr>
<td>Liquid density, $\rho_0$</td>
<td>870 kg/m$^3$</td>
</tr>
<tr>
<td>Speed of sound, $c_0$</td>
<td>1400 m/s</td>
</tr>
<tr>
<td>Kinematic viscosity, $\nu_0$</td>
<td>$8 \cdot 10^{-5}$ m$^2$/s</td>
</tr>
<tr>
<td>Inner radius of pipe, $R$</td>
<td>$6.0 \cdot 10^{-3}$ m</td>
</tr>
</tbody>
</table>

The numbers of elements and modes are chosen so that the number of state variables would be approximately the same. The number of state variables in the modal model is $2n + 1$, where $n$ is the number of modes. In the models A and B (Table 4) $n$ was set to 8, so models had 17 state variables. Riemann windowing is used in both models. The number of state variables in the FEM model with linear interpolation, model C, is $2n-1$, where $n$ is the number of interpolation functions, thus 9 shape functions or correspondingly 8 elements were chosen.

Table 4. Simulation models

<table>
<thead>
<tr>
<th>Name of model</th>
<th>Description</th>
<th>Interpolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model A</td>
<td>Q-model (dissipative), equation (67)</td>
<td>trigonometric</td>
</tr>
<tr>
<td>Model B</td>
<td>Q-model (linear friction), where $\Gamma^2(\tilde{\gamma})$ is approximated as in equation (49b)</td>
<td>trigonometric</td>
</tr>
<tr>
<td>Model C</td>
<td>Dissipative FEM model, equations (43-48)</td>
<td>linear</td>
</tr>
</tbody>
</table>

In the simulation the pipeline is considered to have a steady flow regime for $t < 0$; at $t = 0$ the inflow rate at the left end increases instantaneously by 0.1 l/s and decreases at the right end by the same amount. Since the model is linear it suffices to consider changes from the initial state, i.e. initial pressure can be set to zero. Steady state pressure drop can be verified when all oscillations have been damped.

Comparing different friction models, it is evident (Figure 5) that wave attenuation and distortion of pressure response is much stronger in the dissipative model than in the linear resistance model.
Figure 5. Pressure response of step input in the beginning of pipe.

Comparing the windowed modal model and the FEM model, it is evident (Figure 6) that the FEM model suffers from numerical oscillation problems. Spatial averaging, as described in section 4.2, gives some improvement (Figure 7) but is not so effective as with modal model. Similar results were also found using piecewise cubic Hermite polynomials (cubic splines) as basis.

Figure 6. Pressure response of step input in the beginning of the pipe.
6 Extension to Non-Linear Transmission Lines

In order to include turbulence effect into transmission line modelling it is convenient to assume that friction is proportional to the square of mean velocity. One-dimensional non-linear equations without convective terms can be written:

\[ P_t + \rho_0 c_0^2 U_x = 0 \]
\[ U_t + \frac{1}{\rho_0} P + \xi \frac{U|U|}{4 r_0} = 0 \]  \hspace{1cm} (73a,b)

where \( \xi \) is a dimensionless coefficient of friction that depends on velocity \( U \). Eliminating pressure \( P \) from equations (73a-b) gives a non-linear partial differential equation (PDE) with flow rate \( Q \) as dependent variable:

\[ Q_{tt} + \frac{1}{4 \pi r_0^3} (\xi Q|Q|) - c_0^2 Q_{xx} = 0 \]  \hspace{1cm} (74)

Non-dimensional friction coefficient \( \xi \) can be determined by the semi-empirical Blasius formula

\[ \xi = \frac{0.3164}{Re^{0.25}} \text{, when } 2320 < Re < 10^5 \]  \hspace{1cm} (75)

where the Reynolds number \( Re \) is \( 2 |Q|/(\pi r_0 v_o) \). The coefficient \( \xi \) for laminar flow is \( 64/Re \), then the term \( \xi |Q|/(4 r_0) \) is equal to \( \varepsilon/T \) that gives the linear PDE.
We consider Dirichlet boundary conditions

\[ Q(0) = Q_0, \quad Q(L) = -Q_0 \]  
(76)

The variational form of the equation (74) with Dirichlet boundary conditions (76) is to find \( Q(x) \in H^1(0, L) \) that satisfies the b.c (76) such that

\[
\int_0^L \left( \frac{\partial^2 Q}{\partial x^2} + \frac{1}{4\pi^2} \xi(Q|Q)|, \delta Q + c_0 Q, \delta Q_x \right) dx = 0, \quad \forall \delta Q \in H^1_0(0, L)
\]
(77)

The variational form (77) may be simplified by keeping the term \( \xi|Q| \) constant for each time step, thus flow rate \( Q \) can be solved directly without iteration. We choose a basis from trigonometric functions \( \psi_j \in H^1_0(0, L) \) such that the Ritz approximation satisfies the boundary conditions

\[
\tilde{Q}(x) = Q_0 \frac{L-x}{L} - Q_0 \frac{x}{L} + \sum_{j=1}^n q_j \sin \left( \frac{j\pi x}{L} \right)
\]
(78)

Substituting the Ritz approximation into equation (78) and simplifying yields the ordinary differential equation system of second order

\[
M\ddot{q} + C\dot{q} + Kq = \tilde{f}_1 + \tilde{f}_2
\]
(79)

where

\[
(M)_{ij} = \frac{L}{2} \delta_{ij}, \quad (K)_{ij} = \frac{(j\pi)^2}{2L} c_0^2 \delta_{ij}
\]

\[
(C)_{ij} = \frac{1}{4\pi^2} \int_0^L \xi|Q| \sin \left( \frac{i\pi x}{L} \right) \sin \left( \frac{j\pi x}{L} \right) dx
\]

\[
\left( \tilde{f}_1 \right)_j = (-1)^{i+1} Q_0 \frac{L}{i\pi} - \frac{Q_0}{i\pi}
\]

\[
\left( \tilde{f}_2 \right)_j = \frac{Q_0}{4L\pi^2} \int_0^L \xi|Q| \sin \left( \frac{i\pi x}{L} \right) dx - \frac{Q_0}{4L\pi^2} \int_0^L \xi|Q|(L-x) \sin \left( \frac{i\pi x}{L} \right) dx
\]

(80)

Equations (73a-b) do not include frequency dependent friction effect, since the equations are one-dimensional. Equations (79-80) can be modified to include two-dimensional effects, if we assume that small flow rate and pressure perturbations for turbulence flow behave the same way as for laminar flow. Comparing Q-model (67) and equations (79-80) we can modify matrices as follows

\[
(K)_{ij} = \frac{\omega^2}{2L} c_0^2 \delta_{ij}, \quad \alpha_j = j\pi
\]

\[
(C)_{ij} = \frac{1}{4\pi^2} \int_0^L \left( \xi|Q| + \frac{\varepsilon_j - \varepsilon}{T} \right) \sin \left( \frac{i\pi x}{L} \right) \sin \left( \frac{j\pi x}{L} \right) dx
\]

(81)
For laminar flow where $\xi [Q] / (4\pi R^4)$ is $\epsilon/T$ equation (81) gives the same matrices as in the $Q$-model (67). Pressure can be obtained from equation (73a) when flow rate $Q$ is known.

The damping matrix $C$ and vector $f_2$ have to be calculated numerically since $\xi [Q]$ varies over the pipe length. The damping matrix $C$ is dominated by diagonal terms thus computations can be simplified and sped up by calculating only these diagonal terms. These calculations we have done with Romberg integration because the friction coefficient $\xi$ is discontinuous in the transition area between laminar and turbulent flow.

In the simulation the pipeline (Table 3) is considered to have a steady flow regime for $t < 0$; at $t = 0$ the inflow rate at the left end increases instantaneously by $2.0 \text{l/s}$ and decreases at the right end by the same amount, hence the Reynolds number is about 2650 in steady state. Numerical results are shown in Figure 8 where Riemann windowing is used for the all models.

![Figure 8](image_url)

**Figure 8.** Pressure response of step input in the beginning of the pipe.

7 Conclusions

The modal method has long been recognised as an accurate, convenient, and numerically stable way to model fluid transmission lines [Watton & Tadmori, 1988]. In this paper the method is shown to be a variational method, closely related to finite element methods. Results from the theory of variational methods can be brought to bear. For example, it is known that trigonometric interpolation models converge uniformly to analytic periodic functions [Gottlieb & Orszag, 1977]. Convergence is not so good when the input is not smooth, but as our numerical results show, the modal model with attenuation factors ("windows") can give a much better approximation than finite element models.
Nomenclature

\( B(\cdot) \) \quad \text{Bilinear form}
\( b_n, b_1, b_2 \) \quad \text{Steady state correction factors}
\( C \) \quad \text{Damping matrix}
\( C \) \quad \text{Set of complex numbers}
\( c_0 \) \quad \text{Mean speed of sound in line}
\( f, f_1, f_2 \) \quad \text{Forcing vectors}
\( H_a \) \quad \text{Antisymmetric transfer function}
\( H_s \) \quad \text{Symmetric transfer function}
\( I(\cdot) \) \quad \text{Energy functional}
\( \Im \) \quad \text{Imaginary part}
\( i \) \quad \text{Index, } i=1,2,\ldots,n
\( J_0 \) \quad \text{Zero order Bessel function of first kind}
\( J_1 \) \quad \text{First order Bessel function of first kind}
\( j \) \quad \text{Index, } j=1,2,\ldots,n
\( K \) \quad \text{Stiffness matrix}
\( l(\cdot) \) \quad \text{Linear form}
\( L \) \quad \text{Line length}
\( L(0,L) \) \quad \text{Lebesgue space on } (0,L)
\( l_e \) \quad \text{Length of element}
\( M \) \quad \text{Mass matrix}
\( n \) \quad \text{Number of modes (even) or interpolation functions}
\( P_0 \) \quad \text{Pressure in the beginning of line}
\( P_1 \) \quad \text{Pressure at the end of line}
\( P_a \) \quad \text{Antisymmetric part of pressure}
\( P_s \) \quad \text{Symmetric part of pressure}
\( P \) \quad \text{Pressure}
\( p \) \quad \text{Nodal pressure vector}
\( \bar{P}, \overline{P} \) \quad \text{Approximation and average of } P
\( Q \) \quad \text{Flow rate}
\( Q_0 \) \quad \text{Flow rate in the beginning of line, inflow positive}
\( Q_1 \) \quad \text{Flow rate at the end of line, inflow positive}
\( \dot{Q} \) \quad \text{Trial solution of flow rate}
\( q \) \quad \text{Nodal flow rate vector}
\( q_j \) \quad \text{Flow rate component}
\( \Re \) \quad \text{Real part}
\( r \) \quad \text{Radial co-ordinate}
\( r_0 \) \quad \text{Pipe radius}
\( s \) \quad \text{Laplace variable}
\( \tilde{s} \) \quad \text{Normalised Laplace variable}
\( T \) \quad \text{Wave time}
\( t \) \quad \text{Time}
\( U \) \quad \text{x velocity component}
\( w_j \) \quad \text{Window function component}
\( x \) \quad \text{Co-ordinate along line}
\( Z_0 \) \quad \text{Series impedance}
\( \Gamma^2 \) \quad \text{Propagation operator, equation (49a,b,c)}
\( \varepsilon \) \quad \text{Dimensionless friction coefficient}
\( \varepsilon_i \) \quad \text{Modal damping coefficient}
\( \kappa \) \quad \text{Variable}
\( \nu_0 \) \quad \text{Mean kinematic viscosity}
\( \rho_0 \) \quad \text{Mean fluid density}
\( \xi \) \quad \text{Non-dimensional friction coefficient}
\( \psi_j \) \quad \text{Interpolation function}
\( \omega_i \) \quad \text{Modal natural frequency coefficient}

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