Dynamic Simulations of Flexible Hydraulic-Driven Multibody Systems using Finite Strain Beam Theory

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Abstract

Hydraulically driven mechanisms give rise to a combined flexible multibody system and fluid circuit system. In this work the flexible multibody system is modeled using the unconstrained extremum principle of Simo and Vu-Quoc. This is based on large displacement and rotation theory and yields a non-linear ODE of second order with constant mass matrix. The combined system is solved by a single $L$-stable implicit Runge-Kutta ODE integrator.

Keywords: Dynamic Simulation, Multibody, Hydraulic-Driven Flexible Mechanism.

1 Nomenclature

$A$ Cross section area

$B$ Bulk modulus of fluid

$b_m$ Steady state correction factor

$C_t$ Tangential damping matrix

$C_{turb}$ Orifice coefficient

$D$ Diameter

$c_0$ Mean speed of sound in line

$c_v$ Viscous friction coefficient

$GA_s$ Shear stiffness

$EA$ Axial stiffness

$EI$ Flexural stiffness

$f(t)$ Forcing vector

$H$ Transfer function

$i$ index, $i=1,2,...,n$

$K$ Stiffness matrix

$L$ Line or beam length

$L_2(0,L)$ Lebesgue space on $(0,L)$

$M$ Mass matrix

$M(\xi)$ Bending moment

$N$ Number of nodes

$N(\xi)$ Normal force

$n$ Cross section normal vector

$n$ Number of modes (even)

$P$ Pressure, $(p_r,p_i)^T$

$\tilde{P}$ Trial solution of pressure

$Q$ Flow rate vector, $(Q_r, Q_i)^T$

$Q$ Flow rate

$Q(\xi)$ Shear force

$R$ Reynolds number

$r_0$ Inner pipe radius

$s$ Laplace variable

$\tilde{s}$ Normalised Laplace variable

$T$ Wave time

$t$ Parallel vector of cross section

2 Introduction

There are three main approaches to dynamic analysis of flexible multibody systems:

1. **Inverse dynamic method.** In this method the rigid multibody system is first solved to give initial values, [1]. The flexible system is then solved by iteration with kinematics constraint equations. This is the method used in the ADAMS program.

2. **Constrained extremum principle.** This is the most popular method for solving flexible multibody systems. Individual equations of motions are formulated for each flexible body; equations are then added for the kinematic constraints at the joints. Using a Hamiltonian principle it can be shown that this approach is a constrained extremum problem; such problems are usually solved by a Lagrange multipliers and/or penalty function method. A physical interpretation of the penalty function method is the contact dynamics approach. The choice of co-ordinate (configuration) system and solution procedure varies in different papers [2], [3].

3. **Unconstrained extremum principle.** This is based on large displacement and rotation theory, and the deformation of each body is expressed directly with respect to a single fixed inertial frame using nodal variables, [4] and [5]. The assembling of mass, stiffness and damping matrices includes constraint equations of rotational joints.

The biggest drawback of method 2 is that it yields a highly coupled non-linear differential-algebraic equation (DAE) system, which means that the configuration space contains a hypersurface on which the solution must lie. This is a difficult numerical problem. Another disadvantage is that 3-dimensional multibody systems are significantly harder to solve than 2D-systems.
Method 3, on the other hand, yields a non-linear ordinary differential equation (ODE) system of second order: \( \mathbf{M} \ddot{\mathbf{d}} + \mathbf{r}(d, d) = f(t) \), where the mass matrix \( \mathbf{M} \) is constant. Standard structural dynamics solvers can then be used. Another advantage of this method is that 3D multibody systems are not much harder to solve than 2D systems. Method 3 is the approach used in this work.

In this work the flexible multibody and fluid circuit systems are solved together by a single ODE integrator, so that information from boundaries of different fields is synchronised. An \( L \)-stable implicit Runge-Kutta method is used, because it is better suited for solving stiff ODEs with oscillatory modes than Gear’s method.

Numerical examples of flexible hydraulic-driven multibody systems are presented.

### 3 Pipe Line Models

The authors have developed ODE models for pressure and flow in pipelines [6]. This model is briefly summarized here. The Laplace transformed PDEs describing the dynamics of a viscous compressible fluid in a circular transmission line are:

\[
-L^2 \frac{d^2 P(x)}{dx^2} + \Gamma^2(\bar{s}) P(x) = 0 \quad x \in (0, L)
\]

where \( \Gamma^2(\bar{s}) \) is the propagation operator and \( \bar{s} \) is the normalised Laplace variable (\( \bar{s} = Ts \) where \( T = L/c_o \)).

It has been assumed in equations (1) that the flow is laminar, the fluid is Newtonian, the pipe walls are rigid, and non-linear acceleration term are small.

Consider the Neumann boundary conditions

\[
P'(0) = -\frac{Z_o \Gamma^2(\bar{s})}{L\bar{s}} Q_o, \quad P'(L) = \frac{Z_o \Gamma^2(\bar{s})}{L\bar{s}} Q_i
\]

where \( Z_o = \frac{\rho_o c_o}{\pi r_o^2} \) and inflow is considered as positive.

The variational (or weak) formulation of equation (1) with Neumann boundary conditions (2) is found by multiplying equation (1) by the test function \( \delta P \), integrating over the domain and integrating by parts, yielding

\[
\int_0^L \left( L^2 P' \delta P' + \Gamma^2(\bar{s}) P \delta P \right) dx = LZ_o \frac{\Gamma^2(\bar{s})}{\bar{s}} \left( Q_o \delta P + Q_i + \delta P_o \right)
\]

The variational formulation of the problem is to find the function \( P(x) \) from the Sobolev space

\[
H^l(0, L) = \{ P = P(x) | P, P' \in L_2(0, L) \}
\]

such that (3) holds for all \( \delta P \in H^l(0, L) \).
We solve the Neumann problem (1-2) using trigonometric functions as basis
\[ \{ \psi_j \} = \{ \cos(j \pi x / L) \} \subset H'(0, L) \] for the Ritz approximation:
\[ \tilde{P}(x) = \sum_{j=0}^{n} p_j \cos \left( \frac{j \pi x}{L} \right) \] (5)

Separating pressures in the beginning of the line \( P_0 \) and pressure in the end of line \( P_1 \) into a symmetric part \( P_s \) and an antisymmetric part \( P_a \) and making the same change of variables for flow rates \( Q_0 \) and \( Q_1 \) gives
\[
\begin{align*}
P_s &= (P_0 + P_1) / 2, & Q_s &= (Q_0 + Q_1) / 2 \\
P_a &= (P_0 - P_1) / 2, & Q_a &= (Q_0 - Q_1) / 2
\end{align*}
\] (6)

Substituting approximation (5) into the weak form of Neumann problem (3) and using equations (6), symmetric and antisymmetric pressures can be written in the form
\[
\begin{align*}
P_s &= Z_h H_s Q_s, & P_a &= Z_h H_a Q_a,
\end{align*}
\] (7)

where approximated symmetric and antisymmetric transfer functions \( H_s \) and \( H_a \) are
\[
\begin{align*}
H_s &\approx \frac{2}{\delta} + \sum_{i=2, 4, \ldots}^{n} \frac{4w_i \delta}{\delta^2 + \delta_i \delta + \omega_i^2}, & H_a &\approx \sum_{i=1, 3, \ldots}^{n} \frac{4w_i \delta}{\delta^2 + \delta_i \delta + \omega_i^2},
\end{align*}
\] (8)

In the numerators of the transfer function (8) the propagation operator \( \Gamma^2(\tilde{s}) \) has been approximated by
\[ \Gamma^2(\tilde{s}) + (i \pi)^2 \approx \delta^2 + \delta_i \delta + \omega_i^2, \] (9)

where modal natural frequency coefficient \( \omega_i \) and modal damping coefficient \( \delta_i \) are
\[ \omega_i = i \pi - \frac{1}{4} \sqrt{i \pi \delta} + \frac{1}{16} \delta, & \delta_i = \frac{1}{2} \sqrt{i \pi \delta} + \frac{7}{16} \delta \] (10)

Dimensionless friction coefficient \( \delta \) and steady state correction factor \( b_n \) and window coefficient \( w_i \) are
\[
\delta = \frac{8\nu_0 L}{r_0^2 c_0}, & b_n = \left( 8 \sum_{i=1, 3, \ldots}^{n} w_i / \omega_i^2 \right)^{-1}, & w_i = \frac{\sin(i \pi/(n+1))}{i \pi/(n+1)}
\] (11)

The window coefficients \( w_i, i = 1, 2, \ldots, n \) are added to the transfer functions in order to reduce numerical oscillation (Gibbs phenomenon)

The fluid transmission line model can be written also in state space form:
\[
\begin{align*}
\dot{x} &= Ax + BQ \\
P &= Cx
\end{align*}
\] (12)

where input vector \( Q \), output vector \( P \), state variable vector \( x \) and state space matrices are
\[
Q = \begin{pmatrix}
Q_0 \\
Q_1
\end{pmatrix}, \quad P = \begin{pmatrix}
P_0 \\
1
\end{pmatrix}, \quad x = \begin{pmatrix}
p_0 \\
r_1 \\
\vdots \\
r_n \\
p_n
\end{pmatrix}
\]

\[
A = diag(0, A_1, A_2, \ldots, A_n), \quad A_i = \begin{pmatrix}
0 & -\frac{\omega_i^2}{T^2} \\
1 & -\frac{\varepsilon_i}{T}
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
B_0 \\
B_1 \\
\vdots \\
B_n_{2n+1\times2}
\end{pmatrix}, \quad B_0 = \frac{1}{T}(1 \ 1)
\]

\[
B_{2i-1} = \frac{2w_{2i-1}}{T^2} \begin{pmatrix}
\varepsilon b_n & -\varepsilon b_n \\
T & -T
\end{pmatrix}
\]

\[
B_{2i} = \frac{2w_{2i}}{T^2} \begin{pmatrix}
\varepsilon & \varepsilon \\
T & T
\end{pmatrix}
\]

\[
C = Z_0 \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots \\
1 & 0 & -1 & 0 & 1 & 0 & -1 & \ldots
\end{pmatrix}_{2\times2n+1}
\]

It should be noted that all of these state matrices are constants and that matrix \(A\) is also the Jacobian matrix \((\partial f(x, u)/\partial x)\) of the ordinary differential equation.

### 4 Orifice Model

Next we briefly introduce the model for the flow rate through an orifice when pressure drop is known. Orifices usually operate in the turbulent régime, where flow rate is proportion to the square-root of pressure loss.

The flow function does not satisfy a Lipschitz condition in the neighbourhood of zero pressure difference, so standard uniqueness theorems cannot be applied there. The non-Lipschitz flow function evidently also causes ODE solvers' time-step adjusting algorithms to fail. The flow rate function has to be modified such that it gives a linear pressure–flow rate relation in the neighbourhood of zero pressure difference i.e. in laminar flow, and smooth transition from laminar to turbulent flow. A modified orifice model can be written [7]:

\[
Q_o(\Delta P_o) = \begin{cases}
C_{wrb}A_o\sqrt{\frac{2|\Delta P_o|}{\rho_0}} \operatorname{sgn}(\Delta P_o) & \text{when } |\Delta P_o| > P_u \\
\frac{3A_o v_o R_o}{4D_o} \left(\frac{\Delta P_o}{P_u}\right) \left(3 - \frac{|\Delta P_o|}{P_u}\right) & \text{when } |\Delta P_o| \leq P_u
\end{cases}
\]

where the transition pressure difference \(P_u\) occurs at
\[
P_{tr} = \frac{9R_{tr}^2 \rho_0 \nu_0^2}{8C_{\text{turb}}^2 D_O^2}
\]

(15)

This corresponds to a Reynolds number of \(\frac{3}{2} R_{tr}\). The Reynolds number is defined proportional to orifice diameter \(D_O\)

\[
R = \frac{QD_O}{A_O \nu_0}
\]

(16)

where \(A_O\) is orifice cross section area. Transition Reynolds number \(R_{tr}\), orifice coefficient \(C_{\text{turb}}\), orifice diameter \(D_O\), kinematic viscosity \(\nu\) and fluid density \(\rho_0\) are given parameters.

This orifice model can be used to connect different pipes together or to connect a pipe to a volume. Pipe and volume models have flow rate as inputs and pressures as outputs and correspondingly orifice models have pressure difference as input and flow rate as output, so this connection is possible.

Connecting a pipe to an orifice has some disadvantages since the orifice model (14) is purely algebraic feedback. This will cause the system jacobian matrix to fill with non-zeros and to be non-linear (Figure 1a). This can be avoided by differentiating equation (14) respect time, yielding

\[
\dot{Q}_O(\Delta P_O) = \begin{cases} 
\frac{C_{\text{turb}}A_O}{\sqrt{2 \rho_0 |\Delta P_O|}} \Delta \dot{P}_O & \text{when } |\Delta P_O| > P_{tr} \\
\frac{3A_O \nu_0 R_{tr}}{4D_O} \left(3 - \frac{2|\Delta P_O|}{P_{tr}}\right) \Delta \dot{P}_O \frac{\Delta P_O}{P_{tr}} & \text{when } |\Delta P_O| \leq P_{tr}
\end{cases}
\]

(17)

This modification increases the number of state variables by one but preserves the jacobian's sparsity (Figure 1b). Also, the part of the jacobian that comes from the pipe model remains constant and thus needs to be calculated only once during simulation.

![Figure 1](image-url)

Figure 1. Jacobian of system where 4-mode pipe model and orifice have been connected together. (a) jacobian using formula (14), and (b) jacobian using modified formula (17).
Since we also need pressure time derivatives in equation (17) state space equation (12) can be rewritten

\[ \dot{x} = Ax + BQ \]
\[ \begin{pmatrix} p \\ \dot{p} \end{pmatrix} = \begin{pmatrix} C & O \\ CA & CB \end{pmatrix} x + \begin{pmatrix} O \\ CB \end{pmatrix} Q \]  

(18)

5 Curved Beam Model – 2D Case

The derivation of the beam model presented by Simo and Vu-Quoc [4,5] is rather lengthy: they first derive the partial differential equations of motion as the Euler-Lagrange equations of a Hamiltonian principle, then apply a Galerkin discretization to compute the stiffness and mass matrices and restoring force vector. Here we present a simpler derivation based on the principle of virtual work. This approach is used in finite element literature [8,9]. Our model uses Reissner's beam with initial curvature. This beam, which includes shear strain effects, is based on large displacement and rotation theory. The deformation of each body is expressed directly with respect to a single fixed inertial frame using nodal variables.

Here we use a formulation that relates displacements, strains and virtual work to the initial geometry. This initial geometry is conventionally constructed by dividing the beam into isoparametric elements. The beam kinematic assumption is that material points in the cross section of the undeformed configuration remain plane in the deformation state (Figure 2).

![Figure 2](image.png)

Figure 2. Initial geometry and deformed state, where cross section remains plane.

The initial undeformed neutral axis of the beam can be expressed using interpolation functions

\[ x = \sum_{i=1}^{N} \psi^i(\xi)x^i, \quad y = \sum_{i=1}^{N} \psi^i(\xi)y^i \]  

(19)
where \((x', y')\) is the initial location of the \(i\)th node and \(N\) is the total number of nodes. The same interpolation functions \(\psi^i(\xi)\) are used for all three nodes. These interpolation functions can be taken as second degree Lagrange polynomials, where abscissa \(\xi \in [-1, 1]\) (Figure 3). This selection yields three nodes per element and nine nodal variables per element.

![Figure 3. Interpolation functions in master element](image)

Since the beam model includes shear deformation we have three independent nodal variables per node: displacement at node \(i\) in \(X\)-direction \(u^i\), displacement in \(Y\)-direction \(v^i\), and rotation \(\varphi^i\) that is positive counterclockwise. For isoparametric elements interpolation is the same for displacement and for geometry, thus interpolation for displacements can be written

\[
\begin{align*}
  u(\xi, t) &= \sum_{i=1}^{N} \psi^i(\xi)u^i(t), \\
  v(\xi, t) &= \sum_{i=1}^{N} \psi^i(\xi)v^i(t), \\
  \varphi(\xi, t) &= \sum_{i=1}^{N} \psi^i(\xi)\varphi^i(t)
\end{align*}
\] (20)

Next we introduce finite strain measures that are invariants under rigid body motion. These strain measures are related to for engineering strain that measures change of length relative to the initial length. Axial strain can be written

\[
\varepsilon = \frac{1}{\alpha} \n^T \frac{d(x + u)}{d\xi} - 1 = \frac{1}{\alpha} \n^T \sum_{i=1}^{N} \psi^i(\xi)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) - 1
\] (21)

where \(u=(u, v)^T\) is the displacement vector, \(x=(x, y)^T\) is the location vector and \(n=(\cos\theta, \sin\theta)^T\) is the cross section normal (Figure 2). The length parameter \(\alpha\) is the jacobian of the element mapping and is given by

\[
\alpha^2 = \left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2
\] (22)

The length parameter \(\alpha\) is relative to the initial geometry and thus gives a fixed value. In order to ensure zero axial strain for the initial geometry we have to specify the current rotation angle \(\theta = \theta^m + \varphi\), where the initial angle \(\theta^m\) satisfies

\[
\sin \theta^m = \frac{dy/d\xi}{\alpha}, \quad \cos \theta^m = \frac{dx/d\xi}{\alpha}
\] (23)

The shear strain measure is correspondingly
\[
\gamma = \frac{1}{\alpha} t^T \frac{d(x + u)}{dx} = \frac{1}{\alpha} t^T \sum_{i=1}^{N} \psi_i'(\xi) \begin{pmatrix} x' + u' \\ y' + v' \end{pmatrix}
\]  

where vector \( t = (-\sin \theta, \cos \theta)^T \) is orthogonal to the normal vector \( n \).

Because bending moment and curvature are effectively the same, the strain measure for bending can be given as

\[
\chi = \frac{1}{\alpha} \sum_{i=1}^{N} \psi_i'(\xi) \varphi_i
\]  

Strain measures (21,24 and 25) can be collected together in vector form as

\[
\begin{pmatrix} \varepsilon \\ \gamma \\ \chi \end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix} (x_u + u)\cos \theta + (y_v + v)\sin \theta - \alpha \\ -(x_{\xi} + u)\sin \theta + (y_{\xi} + v)\cos \theta \end{pmatrix} \begin{pmatrix} \varphi \end{pmatrix}
\]  

These strain measures are the same as in [4] or in [8].

Taking the first variation of these strains at an arbitrary point \( \xi \) on the element gives

\[
\alpha \delta \varepsilon = \alpha \begin{pmatrix} \delta \varepsilon \\ \delta \gamma \\ \delta \chi \end{pmatrix} = \sum_{i=1}^{3} \begin{pmatrix} \psi_i'(\xi) \cos \theta & \psi_i'(\xi) \sin \theta & \alpha \psi_i'(\xi) \gamma \\ -\psi_i'(\xi) \sin \theta & \psi_i'(\xi) \cos \theta & -\alpha \psi_i'(\xi)(1+\varepsilon) \\ 0 & 0 & \psi_i'(\xi) \end{pmatrix} \begin{pmatrix} \delta u' \\ \delta v' \\ \delta \varphi' \end{pmatrix}
\]  

or in matrix form

\[
\alpha \delta \varepsilon = \begin{pmatrix} B' & B^2 & B^3 \end{pmatrix} \begin{pmatrix} \delta d^1 \\ \delta d^2 \\ \delta d^3 \end{pmatrix} = B \delta d
\]  

where \( B \) is the kinematic matrix and \( \delta d = (\delta u^1, \delta v^1, \delta \phi^1, \delta u^2, \delta v^2, \delta \phi^2, \delta u^3, \delta v^3, \delta \phi^3)^T \) is the element nodal displacement vector.

Using virtual work principle for a single element in dynamic case it is possible to derive element stiffness and mass matrices, since in equilibrium any admissible virtual displacement gives no work. Virtual work can be given by subtracting external virtual work from internal virtual work (including inertial terms) i.e.

\[
\delta W = \int_{-l}^{l} \left( (\Psi \delta d)^T I_A \Psi \delta d \right) \alpha \delta \varepsilon \, d\xi + \int_{-l}^{l} \begin{pmatrix} N(\xi) \\ Q(\xi) \\ M(\xi) \end{pmatrix}^T \delta \varepsilon \, d\xi - f^T \delta d = 0
\]  

for all admissible virtual displacements \( \delta u, \delta v, \delta \phi \in H^1(0, L) \times C^w(0, \infty) \). The symbols in (29) defined as follows. \( \Psi \) is the element interpolation matrix. The diagonal matrix \( I_A \) consists of cross
section inertial constants: \( I_A = diag(pA, pA, \rho I_A) \), where \( A \) is the cross section area and \( I_A \) is the second moment of cross section area. \( N(\xi), Q(\xi) \) and \( M(\xi) \) represent normal force, shear force and bending moment in cross section at point \( \xi \). Vector \( f^T = (f^i, f^s, m^1, f^i, f^s, m^2, f^i, f^s, m^3) \) is the element nodal force vector considered as external forces.

Substituting equation (28) into the virtual work equation (29) yields the equation for element nodal force

\[
f = \int_{-1}^{1} \Psi^T I_A \Psi \alpha d\xi \int_{-1}^{1} B^T \begin{pmatrix} N(\xi) \\ Q(\xi) \\ M(\xi) \end{pmatrix} d\xi = M\ddot{d} + r(d)
\]

(30)

where \( M \) is mass matrix and vector \( r(d) \) represents internal forces that depend on displacements because of the material relation that will be described later in this section.

The tangent stiffness matrix \( K_t \) comes directly after taking first variation of element nodal force vector (30)

\[
\delta f = M\ddot{d} + \int_{-1}^{1} (B^T \delta Q(\xi) + \delta B^T Q(\xi)) d\xi = M\ddot{d} + (K_{te} + K_{so})\delta d
\]

(31)

Using the linear constitutive relation given by

\[
\begin{pmatrix} N \\ Q \\ M \end{pmatrix} = \begin{pmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EI \end{pmatrix} \epsilon = E\epsilon
\]

(32)
yields, after taking first variation and using equation (28),

\[
\begin{pmatrix} \delta N \\ \delta Q \\ \delta M \end{pmatrix} = E\ddot{\epsilon} = \frac{1}{\alpha} EB\ddot{d}
\]

(33)

It should be noted that although strains (21, 24 and 25) can be finite, the constitutive (or material) relation (32) is not realistic for large strains, since cross section quantities \( (EA, GA, \text{ and } EI) \) are assumed to be constants. Now the material part of the stiffness matrix \( K_{te} \) can be written in more convenient form

\[
K_{te} = \int_{-1}^{1} \frac{1}{\alpha} B^T EB d\xi
\]

(34)

The geometric stiffness matrix \( K_{so} \) is
whose submatrices are

\[
\begin{pmatrix}
0 & 0 & -\psi'\psi'\sin\theta \\
0 & 0 & \psi'\psi'\cos\theta \\
-\psi'\psi'\sin\theta & \psi'\psi'\cos\theta & -\alpha(1+\varepsilon)\psi'\psi'
\end{pmatrix}
\]

The geometric stiffness matrix \( K_{ro} \) has an important role in buckling problems where cross section forces \( N(\xi) \) and \( Q(\xi) \) are much stronger than bending moment \( M(\xi) \). In most hydraulic-driven multibody problems, where links are mainly loaded by bending forces, the system material stiffness matrix \( K_{re} \) dominates the total stiffness matrix \( K \) and hence the geometric stiffness matrix \( K_{ro} \) may be neglected.

At this point we have the structural dynamics ordinary differential equations (ODEs) for single elements, equation (31). In a similar manner the ODEs for the whole structure are found using standard element assembling procedure. Velocity dependent velocity damping terms are added to the model using the mass and stiffness matrices (Rayleigh damping). The structure's ODE is:

\[
M\ddot{\mathbf{d}} + \mathbf{r}(\mathbf{d}, \dot{\mathbf{d}}) = \mathbf{f}(t)
\]

where \( M \) is system mass matrix, \( \mathbf{f}(t) \) is the vector of applied forces, and \( \mathbf{r}(\mathbf{d}, \dot{\mathbf{d}}) \) is the vector of restoring forces that includes damping forces also.

The structural equation (37) differs from other multibody formulation in two ways. First, there are no algebraic constraint equations as in conventional Lagrange formulations of links connected by rotational joints. Secondly, the mass matrix is constant. This allows single step ODE integrators to be used, as described in [10,11]. In particular, the band structure of the mass \( M \), stiffness \( K \), and damping matrix \( C = \partial r(\mathbf{d}, \dot{\mathbf{d}})/\partial \mathbf{d} \) can be exploited for efficient computation.

### 6 Model for Hydraulic Cylinder

A hydraulic cylinder can be modelled as a single volume such that there is no pressure gradient in the volume. The differential equation can be written

\[
\dot{P}_c = \frac{Q_c - A_c x_c}{V_m + A_c x_c} B
\]
where the dominator represents cylinder volume, \( Q_c \) is flow rate into the cylinder, \( x_c, \dot{x}_c \) are piston position and speed, respectively, \( A_c \) is cylinder cross section area, \( B \) is fluid bulk modulus, and \( V_{in} \) is initial volume (Figure 4).

\[ f_c = A_c P_c - f_\mu(x_c, \dot{x}_c) \]  

(39)

The force delivered by the hydraulic cylinder is obtained by subtracting friction \( f_\mu \) from pressure-induced force:

The friction force \( f_\mu(x_c, \dot{x}_c) \) may depend on position and speed of the piston. In simulation model \( f_\mu \) as a combination of static and viscous friction:

\[ f_\mu = \begin{cases} \frac{f_s}{\Delta_{cr}} \Delta + c_v \dot{x}_c & \text{if } |\Delta| < \Delta_{cr} \\ f_s \text{sgn}(\Delta) + c_v \dot{x}_c & \text{elsewhere} \end{cases} \]

(40)

where \( f_s \) is maximum static friction, \( c_v \) is viscosity coefficient, \( \Delta \) is static displacement whose origin is updated step by step, and \( \Delta_{cr} \) is the critical displacement to break static friction contact (Figure 4).

7 Assembling Component Models

An ODE system for a hydraulic-driven flexible mechanism (Figure 5) that includes a single hydraulic pipe line, an orifice, a hydraulic cylinder and a beam can be given in form
\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dQ_o}{dt}
\end{pmatrix}
= \begin{pmatrix}
Ax + BQ \\
g_1(\Delta P_o, \Delta \dot{P}_o) \\
g_2(Q_c, x_c, \dot{x}_c) \\
f(t) - r(d, \dot{d}, P_c)
\end{pmatrix}
\]

where functions \(g_1(\Delta P_o, \Delta \dot{P}_o)\) and \(g_2(Q_c, x_c, \dot{x}_c)\) have been presented in (17) and (38), \(r(d, \dot{d}, P_c)\) includes force caused by hydraulic cylinder, and \(f(t)\) is external force vector e.g. forces caused by gravity. Pressure and pressure derivatives at the end of the pipe line, which are used by the orifice model are given by equation (18).

![Figure 5: Hydraulic driven flexible mechanism](image)

The ODE system (41) is stiff, i.e. it has time scales that differ by many orders of magnitude. Eigenmodes of the pipe line are usually faster than bending modes of flexible mechanism. An \(L\)-stable implicit one-step integrator is very effective for such problems, [11]. An \(L\)-stable integrator damps out the numerical high-frequency oscillations. It is also unconditionally stable \((A\)-stable\), that is any size of time step in stable ODE will get bounded response. In this work we use the modified Rosenbrock method, [12].

Rosenbrock method requires jacobian matrix. For system (41) the jacobian is

\[
J = \begin{pmatrix}
A & -B_2 & 0 & 0 & 0 \\
\partial g_1/\partial x & \partial g_1/\partial Q_o & \partial g_1/\partial P_c & \partial g_1/\partial d & \partial g_1/\partial \dot{d} \\
0^T & \partial g_2/\partial Q_o & 0 & \partial g_2/\partial d & \partial g_2/\partial \dot{d} \\
O & 0 & 0 & 0 & I \\
O & 0 & -\partial r/\partial P_c & -K & -C
\end{pmatrix}
\]
where $B_2$ is second column vector of state matrix $B$, equation (18). Total tangential stiffness $K_t$ includes also stiffness caused by friction force $f_o$, equation (40). Piston position $x_c$ and velocity $\dot{x}_c$ depend directly on nodal displacements, velocities and initial geometry, yielding

$$x_c = \left\| (x_{c_1} + u_{c_1}) - (x_{c_2} + u_{c_2}) \right\|_2^2 - L_{c_n}$$

$$\dot{x}_c = n_c^T (u_c - \dot{u}_c)$$

where subscripts $C_0$ and $C_1$ represent lower and upper joint of cylinder (Figure 5), $L_{c_n}$ is initial length of cylinder, and unit normal vector of piston $n_c$ is

$$n_c = \frac{(x_{c_1} + u_{c_1}) - (x_{c_2} + u_{c_2})}{\left\| (x_{c_1} + u_{c_1}) - (x_{c_2} + u_{c_2}) \right\|_2}$$

(43)

(44)

This kind of hydraulic-driven flexible mechanism formulation makes it possible to calculate the whole jacobian matrix analytically and systematically.

8 Numerical Simulation

In the simulation the hydraulic-driven flexible mechanism as in Figure 5 is considered to have the following flow regime; at $t = 0$ the inflow increases linearly to value $Q = 0.002 \text{ m}^3/\text{s}$ at $t = 1 \text{ s}$ and then decreases linearly to zero at $t = 2 \text{ s}$. No external forces like gravity are included. The discretization of the mechanism (Figure 6) includes two beam elements, a total of 13 degrees of freedom. Parameters for the pipe line, the hydraulic cylinder and the orifice are given in Table 1.

![Figure 6. Finite element discretization of beam, initial state](image_url)

The movement of beam is shown in Figure 7, where total angle in steady state is about 40°. In Figure 8 is shown bending moment at node 3 (Figure 6) that has been calculated afterwards using nodal displacements (post-processing procedure). Five (5) seconds simulation takes about 162 seconds (CPU) using Pentium 100 MHz PC-computer. The total number of time steps was 1281.
Table 1. Parameters for the pipe line, the hydraulic cylinder and the orifice

<table>
<thead>
<tr>
<th>Fluid constants</th>
<th>Kinematic viscosity, $\nu_0 = 80 \ \mu m^2/s$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Density, $\rho_0 = 870 \ kg/m^3$</td>
</tr>
<tr>
<td></td>
<td>Sound speed, $c_0 = 1400 \ m/s$</td>
</tr>
<tr>
<td>Pipe line</td>
<td>Pipe length, $L = 20 \ m$</td>
</tr>
<tr>
<td></td>
<td>Inner pipe radius, $r_i = 6 \ mm$</td>
</tr>
<tr>
<td>Orifice</td>
<td>Orifice diameter, $D_0 = 6 \ mm$</td>
</tr>
<tr>
<td></td>
<td>Orifice coefficient, $C_{\text{turb}} = 0.6$</td>
</tr>
<tr>
<td></td>
<td>Transition Reynolds number, $R = 1000$</td>
</tr>
<tr>
<td>Hydraulic cylinder</td>
<td>Cross section area, $A = 0.01 \ m^2$</td>
</tr>
<tr>
<td></td>
<td>Initial volume, $V_{in} = 1 \ dm^3$</td>
</tr>
<tr>
<td></td>
<td>Critical displacement, $\Delta_{cr} = 0.5 \ mm$</td>
</tr>
<tr>
<td></td>
<td>Static force, $f_s = 100 \ N$</td>
</tr>
<tr>
<td></td>
<td>Viscous coefficient, $c_v = 2500 \ N \ s/m$</td>
</tr>
</tbody>
</table>

Figure 7. Movement of flexible mechanism every 0.2 second interval to time $t = 3 \ s$

Figure 8. Bending moment at node 3 (see Figure 6)
9 Conclusions

The flexible hydraulic-driven mechanism has been modeled using the unconstrained extremum principle, where beam model is based on large displacement and rotation theory, giving a non-linear ODE of second order with constant mass matrix. Although the example considered in this work contains only a single beam, this procedure can be directly generalized to multibody systems where links are connected by rotational joints.

Additional research may be accomplished: 3D beam and bar models, flexible (including inertial effect) modelling of framework of hydraulic cylinder, more sophisticated models for cylinder friction force (e.g. hysteresis and pressure depended effects), special elements for prismatic joints (telescope), non-linear elements for play joints.

References


